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Mathematical model of the analytical solution of a nonlinear system of differential equations for calculating the increase in efficiency indicators during the build-up of military efforts

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Abstract. To assess the degree of fulfillment of certain tasks by troops (forces), certain numerical characteristics are used – performance indicators. They are a quantitative measure of the performance of combat missions and are obtained as a result of the use of various methods of operation research. One of the indicators is the growth rates during the build-up of efforts. The procedure for calculating the increment of performance indicators, taking into account the non-linear nature of the process, is unambiguously uncertain. For a linear class of systems, the problem of analytical solution is solved completely, which makes it possible to calculate performance indicators with the subsequent determination of estimated values of their positive growth. For nonlinear systems there is no such completeness of the results, but for a certain class of them it is possible to find an analytical solution. The objectives of the study are formulated as follows. It is necessary to find such a dependence that would relate the number of parties in one analytical expression, taking into account non-linear nature of the process.

The methodical approach is based on the derivation of the analytic relation, which is the general integral of nonlinear systems of differential equations of means. With the help of this relation (where the functional relationship between the indicators of the number of opposing sides is established), the original system is reduced to one nonlinear differential equation of the first order (for the time indicator), which has an analytical solution by means of table integrals. Obtaining two algebraic equations is key to determining the increase in performance indicators, which significantly increases the efficiency of calculations compared to numerical methods.

Keywords: nonlinear and systems, dynamics of averages, analytical relations, algebraic and equation, performance indicators.

Introduction

To assess the degree of performance of combat missions by troops, certain numerical characteristics and indicators of effectiveness are used. They are a quantitative measure of the performance of combat missions and are obtained as a result of the use of various methods of

operation research during the analysis of complex systems (military formations) or processes. The investigated operation (combat operations) can be quantitatively characterized by a certain set of performance indicators, each of which reproduces its specific feature. Among these indicators, one of the most important is the indicator of losses of the group, which should not exceed the specified value depending on the nature of hostilities (operation). Here, from the point of view of assessment, the most interesting case is the case when it is necessary to additionally calculate the change in this indicator in terms of reducing losses over a certain time.

Under such conditions, it is expedient to use the method of dynamics of averages in a nonlinear formulation, which establishes an analytical relationship between the given parameters. A positive effect (reduction of the loss rate and time of task completion) is achieved by increasing efforts (one task is performed by more than one unit) while other conditions remain unchanged.

It should be noted that the analytical solution of this problem is quite relevant in the applied field of research of nonlinear systems of differential equations, especially in the case of their nonlinearity.

Methodological Framework

Among the large arsenal of mathematical models in the applied field of confrontation between the parties, a special place is occupied by Lanchester's analytical models.

The study of such models, which was initiated in the works of their author, was further developed in the form of a generalized Lanchester model:

$$\frac{dA(t)}{dt} = P - aB(t) - cA(t) - kA(t)B(t);$$

$$\frac{dB(t)}{dt} = Q - bA(t) - gB(t) - lB(t)A(t),$$

where is the number of units of groups of parties (or their number); $A(t)$, $B(t)$

$\frac{dA(t)}{dt}$, $\frac{dB(t)}{dt}$ is the dynamics of changes in the number of units of the parties over time;

a, b – the intensity of the effective damaging effect of each unit of the parties, respectively (in the general case of the function of quantities); $A(t)$, $B(t)$

P, Q – the speed of recovery of losses due to the introduction of reserves;

C, G, K, L are non-negative parameters.

An analysis of the generalized Lanchester model at $k=l=0$ was carried out in (Morse and Kimbell, 1956; Chuev 1970; Saati, 1977). For $k\neq0$, $l\neq0$, but at $c=g=0$, it was continued in a monograph (Wentzel, 1968), as well as in a number of other works, for example (Diener, 1970).

The next stage in the development of the method of dynamics of averages in the form of Lanchester equations was the further detailing of the process of counteraction for different types of means with an increase (in terms of the number of these means) of the order of the system of differential equations, which significantly complicated its study. To reduce the dimensionality of the problem in (Pliuta 1989) and in a number of other works (Brezgin, Buravlev, 2011), an approach using the coefficients of proportionality of different types of means was proposed. As a result, the original multidimensional system was transformed into a system of differential equations. Thus, the study of the system of differential equations has not lost its significance to this day.

Results and Discussions

Presentation of the essence of the methodical approach to determining the analytical solution

of the nonlinear system of differential equations of the method of dynamics of averages with the subsequent calculation of the increase in efficiency indicators.

In general, the initial nonlinear system of the method of means dynamics is represented in the form:

$$\begin{aligned}\frac{dm_A}{dt} &= -f_B(\lambda_B)\cdot m_B;\\ \frac{dm_B}{dt} &= -f_A(\lambda_A)\cdot m_A,\end{aligned} \tag{1}$$

where functions are approximated by dependencies of the form: $f_A(\lambda_A)$, $f_B(\lambda_B)$

$$\begin{aligned}f_A(\lambda_A) &= \lambda_A\left(\frac{m_B}{N_B}\right)^q;\\ f_B(\lambda_B) &= \lambda_B\left(\frac{m_A}{N_A}\right)^q.\end{aligned} \tag{2}$$

The objectives of the study are formulated as follows. It is necessary to find such a dependence that would relate the number of sides in one analytical expression, taking into account the nonlinear form of the system (1). To do this, as in the previous case, we will use the methods of finding the general integrals of the system of differential equations. Then writing the equations in the form (1), where $q = 1$ allows you to find one of these integrals. Indeed, performing the necessary mathematical calculations, we get:

$$m_A = \frac{\lambda_B N_B}{\lambda_A N_A} m_B + C, \tag{3}$$

where the constant integration C is defined by the expression:

$$C = \frac{N_A^2\lambda_A - N_B^2\lambda_B}{\lambda_A N_A}. \tag{4}$$

Let us substitute (3) taking into account (4) in the original equation (1) and we get:

$$\frac{dm_B}{dt} = -\frac{\lambda_B}{N_A} m_B^2 - \left(\frac{\lambda_A N_A}{N_B} - \frac{\lambda_B N_B}{N_A}\right) m_B. \tag{5}$$

Let's rewrite (5) as follows:

$$dt = \frac{dm}{-\frac{\lambda_B}{N_A} m_B^2 - \left(\frac{\lambda_A N_A}{N_B} - \frac{\lambda_B N_B}{N_A}\right) m_B}, \tag{6}$$

where the right part is represented by the form of a tabular integral (Bronstein, Semendyaev, 1986).

Then, integrating the expression (6), we get the dependency:

$$t = \frac{1}{-\frac{\lambda_A N_A}{N_B} + \frac{\lambda_B N_B}{N_A}} \ln \frac{m_B}{-\frac{\lambda_B}{N_A} m_B - (\frac{\lambda_A N_A}{N_B} - \frac{\lambda_B N_B}{N_A})} + C_1. \quad (7)$$

Depending on (7), the constant integration of_{C1} is determined from the initial conditions. Finally, for t, the expression (7) is written as:

$$t = \frac{N_A N_B}{\lambda_B N_B^2 - \lambda_A N_A^2} \ln \frac{m_B(-\frac{\lambda_B}{N_A} N_B - \frac{\lambda_A N_A^2 - \lambda_B N_B^2}{N_A N_B})}{N_B(-\frac{\lambda_B}{N_A} m_B - \frac{\lambda_A N_A^2 - \lambda_B N_B^2}{N_A N_B})}. \quad (8)$$

Let's combine (3) and (4) and write:

$$m_A = \frac{\lambda_B N_B}{\lambda_A N_A} m_B + \frac{N_A^2 \lambda_A - N_B^2 \lambda_B}{\lambda_A N_A}. \quad (9)$$

Equations (8) and (9) are a solutionof the nonlinear system (1) and are further used to determine the increase in performance indicators during the increase in efforts.

We will also show another way to solve the system (6), by $q=1$. Then equation (5) is rewritten as:

$$\dot{m}_B + (\frac{\lambda_A N_A}{N_B} - \frac{\lambda_B N_B}{N_A}) m_B = \frac{-\lambda_B}{N_A} m_B^2. \quad (10)$$

The form of notation (10) corresponds to the Bernoulli equation (Bronstein, Semendyaev, 1986) (Dwight, 1977), which has its own solution algorithm. Here, by special substitution of variables, equation (10) is reduced to a linear differential equation:

$$\dot{z} + az = b, \quad (11)$$

which is easy to integrate.

Further, by reverse substitution of variables, the solution of the original system (10) is found:

$$m_B = \frac{N_B(\frac{\lambda_A N_A}{N_B} - \frac{\lambda_B N_B}{N_A})}{\frac{\lambda_A N_A}{N_B} e^{(\frac{\lambda_A N_A}{N_B} - \frac{\lambda_B N_B}{N_A})t} - \frac{\lambda_B N_B}{N_A}}, \quad (12)$$

from which time *t* is determined. Oheld expression coincides with (8) taking into account the analytic transformation of this expression. With a similar approach, the value of m_A is calculated.

Now let's find an increase in performance indicators. Suppose that at the final stage of hostilities (t 1) the value *of* m B has reached the value of m B (t 1) (the parameter at which side B refuses to conduct further hostilities). Then the value *of* $m_A(t_1)$ is uniquely found from expression (9).

Suppose that side A has increased the number of its units *by n* times, while the parameters of side B have remained unchanged. With this in mind, let us set ourselves the goal of determining the reduced time of hostilities (t 2), as well as reducing the value of the relative losses of side *A* at time t_2, assuming that $m_B(t_1) = m_B(t_2)$.

In this case, expressions (13) and (14) are written as:

$$m_A(t_2) = \frac{\lambda_B N_B}{\lambda_A n N_A} m_B(t_1(t_2)) + \frac{(nN_A)^2 - N_B\lambda_B}{\lambda_A n N_A}; \tag{13}$$

$$t_2 = \frac{nN_A N_B}{\lambda_B N_B^2 - \lambda_A n^2 N_A^2} \ln \frac{m_B(t_1(t_2))(-\frac{\lambda_B}{N_A} N_B - \frac{\lambda_A n^2 N_A^2 - \lambda_B N_B^2}{nN_A N_B})}{N_B(-\frac{\lambda_B}{nN_A} m_B(t_1(t_2)) - \frac{\lambda_A n^2 N_A^2 - \lambda_B N_B^2}{nN_A N_B})}. \tag{14}$$

Then the time of hostilities will be reduced by the amount of:

$$\Delta t = t_1 - t_2,$$

and the relative loss of side *A* is reduced by the amount of:

$$\Delta B_A = \frac{N_A - m_A(t_1)}{N_A} - \frac{nN_A - m_A(t_2)}{nN_A}.$$

Thus, the desired effect will be achieved in a shorter period of time and with fewer losses.

The solution of system (1) by *q=2* looks somewhat more complicated. Here it is advisable to use the Bernoulli (Saati) method. Then the equation for m_B is written as:

$$m_B = -\frac{\lambda_A N_A}{N_B^{2+R}} m_B^{2+R}, (R = \frac{\lambda_B N_B^2}{\lambda_A N_A^2}),$$

and its solution will be determined by the expression:

$$m_B = \sqrt[-1-R]{(-1-R)(-\frac{\lambda_A N_A}{N_B^{2+R}})t + N_B^{-1-R}}.$$

Here, the calculation of the increase in performance indicators is similar to the case where *q = 1*.

The basis for choosing a model in the form of (1) was the calculation of its parameters according to the following algorithm. Initially, system (1) was matched with the transient process, which is shown in Fig. 1.

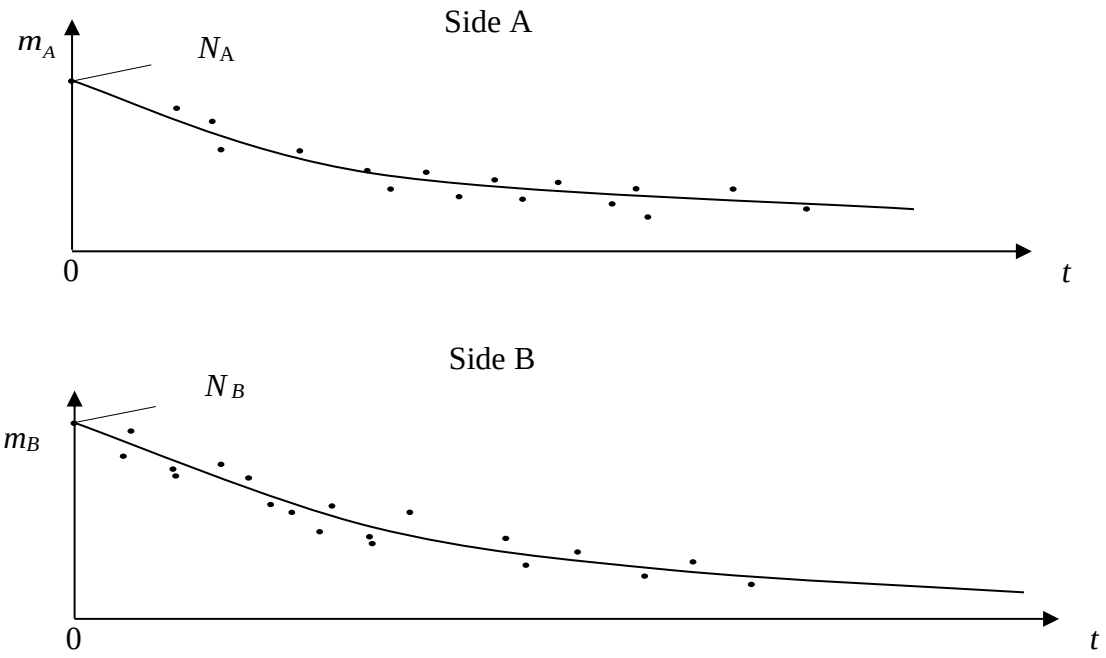


Figure 1. Calculation of model parameters

Fig. 1 dots indicate the values that were obtained by different variants of simulation modeling. Subsequently, for each dependency, a regression polynomial was formed in the form of a dependence:

$$m_{A(B)} = a_0^{A(B)} + a_1^{A(B)} t + a_2^{A(B)} t^2 + a_3^{A(B)} t^3 + \dots \; . \qquad (15)$$

To determine the parameters of this polynomial, the method of least squares was used. Then, by differentiating the dependencies (1, 2), the left-hand sides of the equations (1) were determined. At the end of the whole process, the coefficients and . As a criterion of truth, those values of independent variables were taken, in which over the entire time interval of identification of the deviation of the calculated (pd time of differentiation of equations ($1 \lambda_A \lambda_B$)) values of the corresponding values (found by polynomial dependence) did not exceed the integral error. $m_{A(B)\,розр}$ $m_{A(B)}$ $\Delta_{A,B\,зад}$

A number of works (Romanchenko, Kotlyarov) are devoted to the issues of determining analytical dependencies for performance indicators on the basis of the method of dynamics of averages in a linear formulation, where the solution of equations is carried out on the basis of general analytical approaches to the study of the system of differential equations (Bronstein, 1986, Wentzel, 1972, Dwight, 1977).

For example, in the work (Romanchenko, Kotlyarov, Pavlyuk, 2018), thisplace is occupied by the model of conducting bilateral combat operations of groups in the confrontation of the parties on the basis of linear differential equations of the dynamics of means. In general, they are written in the form of:

$$\begin{aligned} \frac{dm_A}{dt} &= -m_B\lambda_B; \\ \frac{dm_B}{dt} &= -m_A\lambda_A, \end{aligned} \tag{16}$$

where is the number of combat-ready units of the parties' groups; m_A, m_B

$\lambda_A = const$, $\lambda_B = const$ – fire performance of the parties;

$\frac{dm_A}{dt}$, $\frac{dm_B}{dt}$ – the dynamics of changes in the number of combat-ready units of the parties over time.

The solution of the system (16) allows you to establish analytical dependencies. For this purpose, methods of finding the general integrals of the system of differential equations are used. In this case, writing the equations in the form (16) allows you to find one of these integrals. Indeed, performing the necessary mathematical calculations, we get:

$$m_A^2 = \frac{\lambda_B}{\lambda_A} m_B^2 + C, \tag{17}$$

where *C is* an arbitrary integration constant.

Expression (17) is valid for any point in time and, thus, in a general way, establishes an analytical relationship between the numbers of groups, which is especially important when calculating such an indicator of efficiency as the balance of forces of the parties.

As for the constant integration of C, it is unchanged both for the initial balance of forces of the parties and for the numbers of the parties at the end of the operation. This is used in the future. Based on the requirements of regulatory documents, through the values and under the known, such a ratio of forces of the parties at the end of the operation is set, which satisfies us. Then the value of *C* is m_A^k m_B^k λ_A λ_B easily found from equation (17).

Now let's write down this equation for the initial values of the number of sides and in the form: $m_A^н$ $m_B^н$

$$m_A^н = \sqrt{\frac{\lambda_B}{\lambda_A}(m_B^н)^2 + C}. \tag{18}$$

Thus, taking into account the previously found value of C, given the initial composition of the grouping of one of the parties (for example) it is possible to find, according to expression (18), the initial number of the $m_B^н$ other side. It should be emphasized that in this case, the initial balance of forces $\frac{m_A^н}{m_B^н}$ ($m_B^н$ – set, $m_A^н$, – determined) will ensure the achievement of the goal of the operation according to the previously set criterion of the balance of forces at the time of its $m_A^н$ $\frac{m_A^н}{m_B^н}$ $m_B^н$ $m_A^н$

$\frac{m_A^к}{m_B^к}$ completion.

In the considered methodical approach, only the issue of determining the necessary time for conducting an operation (combat operations) according to the selected criterion remains unresolved.

In the paper (Romanchenko, Kotlyarov, Shapoval, 2008) the following algorithm for solving this problem is proposed. First, the system (16) is solved for a given value and a value of . Then the following time dependencies are obtained: m_B^H m_A^H

$$2\sqrt{\lambda_A}m_A = (f_A(\lambda_A, m_A^H) - f_B(\lambda_B, m_B^H)) \cdot e^{\sqrt{\lambda_A\lambda_B}t} + (f_A(\lambda_A, m_A^H) + f_B(\lambda_B, m_B^H)) \cdot e^{-\sqrt{\lambda_A\lambda_B}t};$$
$$2\sqrt{\lambda_B}m_B = -(f_A(\lambda_A^{/}, m_A^H) - f_B(\lambda_B, m_B^H)) \cdot e^{\sqrt{\lambda_A\lambda_B}t} + (f_A(\lambda_A, m_A^H) + f_B(\lambda_B, m_B^H)) \cdot e^{-\sqrt{\lambda_A\lambda_B}t}. \quad (19)$$

Further, in expression (19), the right and left parts of the equations are summed up, after which the value of the time parameter *t* is determined from the obtained dependence in the form:

$$t = \frac{1}{\sqrt{\lambda_A\lambda_B}} \ln \frac{f_A(\lambda_A, m_A^H) + f_B(\lambda_B, m_B^H)}{m_A\sqrt{\lambda_A} + m_B\sqrt{\lambda_B}}. \quad (20)$$

According to the known, by substituting in ($m_{A,B}^H$, $\lambda_{A,B}$ 20) instead of the values of their values that satisfy the goals of achieving the operation, the duration of hostilities t is also determined m_A *i* m_B $m_{A,B}^K$. Note that for linear systems, the problem of analytical solution is solved completely. This makes it possible to calculate other performance indicators with the subsequent determination of the estimated values of their positive growth.

As for nonlinear systems, there is no such completeness of results in this field of research, but for a certain class of them it is possible to obtain an analytical solution.

Conclusions

The proposed methodical approach is based on the derivation of the analytical relation, which is the general integral of the nonlinear systems of differential equations of the dynamics of the means. With the help of this analytic expression (where the functional relationship between the indicators of the number of opposing sides is established), the original system is reduced to one nonlinear differential equation of the first order (for the time indicator), which has an analytic solution by means of tabular integration or by solving the Bernoulli equation. Obtaining two algebraic equations is key to determining the increase in performance indicators.

A further direction of research can be considered a numerical interpretation of the use of this approach to determine the increase in performance indicators in specific operations (combat operations).

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